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Digraphs of degree two which miss the Moore bound by two

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Abstract

It is well known that Moore digraphs do not exist except for trivial cases (degree one or diameter one). Consequently, for a given maximum out-degree d and a given diameter, we wish to find a digraph whose order misses the Moore bound by the smallest possible 'defect'. For diameter two and arbitrary degree there are digraphs which miss the Moore bound by one. No examples of such digraphs of diameter at least three are known, although several necessary conditions for their existence have been obtained. In the case of degree two, it has been shown that there are no digraphs of diameter greater than two and defect one. There are five nonisomorphic digraphs of degree two, diameter two and defect two. In this paper we prove that digraphs of degree two and diameter $k\geqslant 3$ which miss the Moore bound by two do not exist.

Keywords: Directed graphs; Extremal graphs; Moore bound; Defect

1. Introduction

The well-known *degree/diameter problem* for digraphs is to determine, for given positive integers d and k, the largest order $n_{d,k}$ of a digraph of out-degree at most d and diameter at most k. An obvious upper bound on $n_{d,k}$ obtained by counting the possible number of vertices at distance t $(0\leqslant t\leqslant k)$ from a fixed vertex is the *Moore bound* $M_{d,k}$:

$$n_{d,k}\leqslant M_{d,k}=1+d+d^2+\cdots+d^k.$$

As was proved some time ago with the help of spectral methods in [12] or [7], the equality $n_{d,k}=M_{d,k}$ holds only in the trivial cases when $d=1$ (and then the digraphs

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are directed cycles of length $k+1$) or $k=1$ (complete digraphs of order $d+1$). Thus, if $d,k \geqslant 2$ then we always have $n_{d,k} = M_{d,k} - \varepsilon_{d,k}$, where $\varepsilon_{d,k} > 0$. It is natural to refer to the number $\varepsilon_{d,k}$ as the *defect* corresponding to the pair d,k.

The degree/diameter problem for digraphs is obviously equivalent to determining the defect $\varepsilon_{d,k}$ for each pair d,k. However, exact results in this area are surprisingly scarce, and in most cases difficult to prove. An exception occurs for the smallest non-trivial diameter, that is, $k=2$, where the defect $\varepsilon_{d,2}$ is known to be equal to 1 for any $d \geqslant 2$. Indeed, we then have $\varepsilon_{d,2} \geqslant 1$ by the above, and there are examples of digraphs on $n_{d,2} = M_{d,2} - 1$ vertices for any $d \geqslant 2$, namely, the line digraphs of complete digraphs.

In contrast to this, for small degrees it takes a number of pages of subtle arguments to obtain just *bounds* on $\varepsilon_{d,k}$ and no general *exact* results are known so far. More specifically, for the degrees $d=2$ and 3 it was shown in [10,6], respectively, that $\varepsilon_{2,k} \geqslant 2$ for all $k \geqslant 3$. The general question of whether or not $\varepsilon_{d,k} \geqslant 2$ for $d \geqslant 4$ and $k \geqslant 3$ remains completely open. The interested reader is invited to consult [1,3–5,8] for further partial results.

In this paper, we will concentrate on digraphs of out-degree $d=2$ and diameter $k \geqslant 3$; for these values, we have $\varepsilon_{2,k} \geqslant 2$ by Miller and Fris [10]. For convenience, we will say that a digraph of out-degree at most d and diameter at most k on exactly $M_{d,k} - 2$ vertices is a $(2,k)$-*digraph of defect two*. According to the results mentioned above, to prove that $\varepsilon_{2,k} = 2$ for some $k \geqslant 3$ is equivalent to establishing the existence of a corresponding $(2,k)$-digraph of defect two. In the case of the diameter $k=2$, it was shown in [9] that there are exactly five non-isomorphic $(2,2)$-digraphs of defect two. The same paper initiated attacks to exclude the existence of such digraphs of diameter $k \geqslant 3$ by deriving the following interesting necessary condition of arithmetic nature: If $\varepsilon_{2,k} = 2$ for some $k \geqslant 3$ then $k+1$ must be a divisor of $2(2^{k+1} - 3)$, the number of arcs in a $(2,k)$-digraph of defect two. A computer check [9] showed that the only two values in the range $3 \leqslant k \leqslant 10^7$ that fulfil this condition are 274485 and 5035921, so that for all but the two values of k, $3 \leqslant k \leqslant 10^7$ we have $\varepsilon_{2,k} \geqslant 3$. It is not known to us if this divisibility condition is satisfied for infinitely many values of k.

The purpose of this paper is to show that $(2,k)$-digraphs of defect two do not exist for any $k \geqslant 3$, that is, $\varepsilon_{2,k} \geqslant 3$ for $k \geqslant 3$. Our arguments will be completely elementary and independent of the above arithmetic conditions if $k \geqslant 6$.

2. Basic facts

Throughout the rest of this paper, the symbol H will denote a $(2,k)$-digraph of defect 2, that is, with exactly $M_{2,k} - 2 = 2^{k+1} - 3$ vertices. As it was shown in [11], the digraph H must, in fact, be *diregular* (which means that the in-degree and the out-degree of each vertex is 2) and its diameter must be *equal* to k.

We will say that a vertex w' is an *out-neighbour* (*in-neighbour*) of a vertex w of H if ww' ($w'w$) is an arc of H. Instead of out-neighbour we will often use just the term *neighbour*. For $0 \leqslant \ell \leqslant k-1$ we denote by $N_\ell(w)$ the set of vertices of H whose

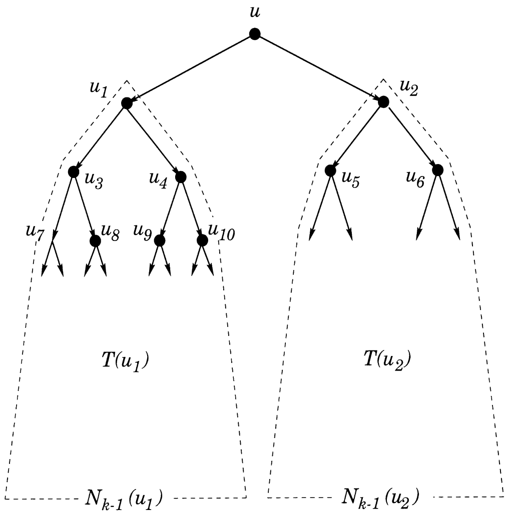

Fig. 1. Picture of a $(2,k)$-digraph of defect two.

distance from w is exactly ℓ. Further, for $0 \leqslant \ell \leqslant k-1$ let $T_\ell(w) = \bigcup_{i=0}^{\ell} N_i(w)$. The set $T_\ell(w)$ thus comprises all vertices at distance at most ℓ from w and represents the 'tree of depth ℓ below the vertex v'. We will make no distinction between $T_\ell(w)$ as a *set of vertices* and as a *subdigraph of H* induced by this set of vertices; no confusion will arise.

A standard counting argument shows that $|N_\ell(w)| = 2^\ell$ for $0 \leqslant \ell \leqslant k-1$ (otherwise H would have less than $M_{2,k}-2$ vertices). It is useful to realize that this is equivalent to claiming that the endvertices of all paths of length at most $k-1$ that emanate from w are all distinct, i.e. there are no two distinct $w \to w'$ paths of length $\leqslant k-1$. In Sections 3 and 4, we will occasionally refer to this fact by saying that *there are no two short* $w \to w'$ *paths*. It also follows that for $0 \leqslant \ell \leqslant k-1$ we have $|T_\ell(w)| = 2^{\ell+1}-1$ for each vertex w of H. Among the sets (or subdigraphs) $T_\ell(w)$ the one most frequently referred to will be $T_{k-1}(w)$; for simplicity, we will in this case omit the subscript and set $T_{k-1}(w) = T(w)$.

Let w be a *fixed* vertex of H. In such a case, we will use the notation $N_1(w) = \{w_1, w_2\}$ for the neighbours of w. When dealing with iterated neighbourhoods we will set $N_1(w_1) = \{w_3, w_4\}$, $N_2(w_2) = \{w_5, w_6\}$; in some cases when a deeper analysis is necessary we will also use the neighbourhoods $N_1(w_3) = \{w_7, w_8\}$ and $N_1(w_4) = \{w_9, w_{10}\}$. In the course of our further exposition we will often consider more than one fixed vertex at a time and the above notation regarding the subscripts will then apply to each of the fixed vertices.

For the reader, it will be helpful to depict the digraph H with a fixed vertex, say, u, as in Fig. 1; the (iterated) neighbours of u are labelled in accordance with our notational convenience.

When dealing with Fig. 1 it is important to realize that the picture itself contains a total of $1+|T(u_1)|+|T(u_2)|=2^{k+1}-1$ vertices. Since H is assumed to be a $(2,k)$-digraph of defect two and so its number of vertices is only $2^{k+1}-3$, it follows that some of the vertices occur in the picture repeatedly — either some vertex occurs three times or two vertices occur two times each. If a vertex w occurs in Fig. 1 three times then, necessarily, $w=u$ (otherwise w would have to appear twice in $T(u_1)$ or $T(u_2)$ which, as we already know, is impossible). But this would mean that H contains a directed cycle of length not exceeding k, which contradicts the result of [9]. Therefore, there must exist two distinct vertices w_1, w_2 that appear in Fig. 1 two times each. In accordance with the terminology introduced in [2] we will call the vertices w_1 and w_2 the *repeats* of the vertex u. The set of repeats of a vertex w will be denoted by $R(w)$. It follows that $|R(w)|=2$ for each vertex w of H; in particular, we have $R(u)=\{w_1,w_2\}$. The repeat w_1 of u necessarily occurs once in $T(u_1)$ and once in $T(u_2)$, and so does w_2. Moreover, w_1 must appear either in $N_{k-1}(u_1)$ or in $N_{k-1}(u_2)$, as otherwise we would have two short $u \to w_1$ paths; of course the same holds true for the repeat w_2. For completeness we note that the repeats w_1 and w_2 of u can also be characterised by the property that they are the only pair of vertices such that, for each $i=1,2$, there exist two $u \to w_i$ paths of length $\leqslant k$ in our digraph H.

We continue by proving that in our $(2,k)$-digraph H of defect two, any two neighbourhoods can share at most one vertex and that the distance 2 neighbourhoods must differ in at least one vertex. In the course of the proof, the reader is invited to use the drawing of Fig. 1.

Lemma 1. *If u and v are distinct vertices of H then $|N_1(u)\cap N_1(v)|\leqslant 1$.*

Proof. Assume that $N_1(u)=N_1(v)$. The vertex v has to appear in the set $T(u_1)\cup T(u_2)$. But as $N_1(u)=N_1(v)$ this means that in our digraph there would be a $v \to v$ walk (and hence a cycle through v) of length not exceeding k, a contradiction. □

Lemma 2. *For any two distinct vertices u, v of H we have $N_2(u)\neq N_2(v)$.*

Proof. Keeping to our standard notation, let $N_1(u)=\{u_1,u_2\}$, $N_1(v)=\{v_1,v_2\}$, $N_1(u_1)=\{u_3,u_4\}$ and $N_1(u_2)=\{u_5,u_6\}$. Suppose further that $N_1(v_1)\cup N_1(v_2)=\{u_3,u_4,u_5,u_6\}$. Without loss of generality, we can assume that $u_3\in N_1(v_1)$. By Lemma 1 we have $u_4\notin N_1(v_1)$, so we may assume that $N_1(v_1)=\{u_3,u_5\}$. Then $N_1(v_2)=\{u_4,u_6\}$.

We know that the vertex u_1 cannot appear in $T(u_3)$ and $T(u_4)$, but u_1 must appear in the set $T(u_5)\cup T(u_6)$. In fact, in order to reach u_1 from the vertices u_2, v_1 and v_2 by paths of length at most k, the vertex u_1 has to appear in *both* $T(u_5)$ and $T(u_6)$.

Observe that the $u_5 \to u_1$ path in $T(u_5)$ cannot contain the vertex u (else there would be a $u \to u$ walk of length not exceeding k), and therefore for the second in-neighbour w of u_1 ($w\neq u$) we have $w\in T(u_5)$. A similar argument applies to the $u_6 \to u_1$ path in $T(u_6)$, and so the vertex w also appears in $T(u_6)$. Note that in both cases we have $w\notin N_{k-1}(u_5)$ and $w\notin N_{k-1}(u_6)$. But then there would be two short $u_2 \to w$ paths in H, a contradiction. □

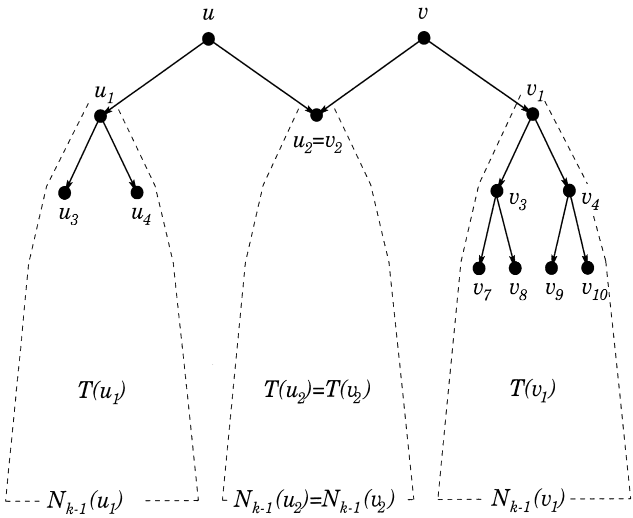

Fig. 2. Picture of a $(2,k)$-digraph of defect 2 with fixed vertices u, v.

As indicated before, we will often consider more than one fixed vertex of H at a time. Most typically, we will pick a pair of fixed vertices u, v in H that have a common neighbour, i.e. $N_1(u) \cap N_1(v) \neq \emptyset$. Following our standard notation where $N_1(u) = \{u_1, u_2\}$ and $N_1(v) = \{v_1, v_2\}$, we will without loss of generality assume that $u_2 = v_2$. In such a case we will depict the situation as in Fig. 2.

We note that Fig. 2 depicts, in fact, two overlapping copies of our $(2,k)$-digraph H of defect two: one copy is induced by the vertex set $\{u\} \cup T(u_1) \cup T(u_2)$ and the other is induced by the set $\{v\} \cup T(v_1) \cup T(v_2)$; the copies overlap in the set $T(u_2) = T(v_2)$. It follows that most of the vertices of $T(u_1)$ must appear in $T(v_1)$ as well, and information about their exact location will be essential in proving our main result.

We will conclude this section with an illustration of the way Fig. 2 can be helpful; we will also use the properties of repeats.

Lemma 3. Let u, v be a pair of distinct vertices of H that have a common neighbour. Then, $v \in N_{k-2}(u_1) \cup N_{k-1}(u_1)$ and $u \in N_{k-2}(v_1) \cup N_{k-1}(v_1)$.

Proof. The two assertions are clearly symmetric, so we concentrate only on the first one. Assume the notation is as above, that is, $u_2 = v_2$. The vertex v must appear in the union $T(u_1) \cup T(u_2) = V(H) \setminus \{u\}$. However, v cannot appear in $T(u_2) = T(v_2)$, and therefore $v \in T(u_1)$. Now, if $v \in T_{k-3}(u_1)$ then we would have $u_2 = v_2 \in T_{k-2}$ and so $N_1(u_2) = \{u_5, u_6\} \subset T_{k-1}(u_1)$. But then u_2, u_5, u_6 would all be repeats of u, contrary to the fact that $|R(u)| = 2$. Consequently, $v \in N_{k-2}(u_1) \cup N_{k-1}(u_1)$. $\square$

3. Stable vertices

We will refer throughout to our $(2,k)$-digraph H of defect two and to the way it is depicted in Fig. 2 (including the notation). As mentioned in the previous section, most of the vertices in $T(v_1)$ appear also in $T(u_1)$; we now concentrate on details about their location in $T(u_1)$. The following concept will play a key role in the analysis. A vertex $w \in T(v_1)$ will be called *stable* if either $w = v_1$, or $w \in N_1(v_1)$ and $T_{k-2}(w)\setminus\{w\}$ contains at most one repeat of the vertex v, or if $w \in N_\ell(v_1)$, $2 \leqslant \ell \leqslant k-3$, and no repeat of v appears in $T_{k-1-\ell}(w)\setminus\{w\}$. In other words, a vertex $w \in T(v_1)$ is stable if, in the pictorial representation of Fig. 2, the number of repeats of v located 'under' the vertex w is either zero or does not exceed $2 - d(v_1, w)$.

Lemma 4. *Let $w \in T(v_1)$ be a stable vertex and let $d(v_1, w) = \ell$ for some ℓ, $0 \leqslant \ell \leqslant k-3$. Assume that w also appears in $T(u_1)$. Then either $d(u_1, w) \leqslant \ell$ or $w \in N_{k-1}(u_1)$.*

Proof. Let $w \in T(u_1)$ and let $m = d(u_1, w)$, $m \leqslant k-1$. Assume that $m > \ell$; our goal is to show that $m = k-1$. To this end, we will examine the set $N_{k-m}(w)$, looking at it from the perspective of both u_1 and v_1. As $\ell + k - m \leqslant k-1$, we see that $N_{k-m}(w)$ is a subset of $T(v_1)$. Note that, by the definition of m, the set $N_{k-m-1}(w)$ is a subset of $N_{k-1}(u_1)$, and therefore $N_{k-m}(w)$ is *not* a subset of $T(u_1)$.

The key observation is that, possibly up to the vertex u and the repeats of the vertex u_1, no other vertex in the set $N_{k-m}(w)$ can appear in $T(u_1) \cup \{u\}$. This is an easy consequence of the properties of repeats, in our case, of repeats of u_1. As the vertex set of our $(2,k)$-digraph of defect two is the set $\{u\} \cup T(u_1) \cup T(u_2)$, it follows that, possibly up to the three vertices mentioned above, all the remaining vertices of $N_{k-m}(w)$ belong to $T(u_2)$. More specifically, setting $I(w) = N_{k-m}(w) \cap T(u_2)$, we have $N_{k-m}(w)\setminus(\{u\} \cup R(u_1)) \subset I(w)$. Using the fact that $|N_{k-m}(w)| = 2^{k-m}$ we obtain $2^{k-m} - 3 \leqslant |I(w)|$.

Now, look at the set $N_{k-m}(w)$ as a subset of $T(v_1)$. Since $T(v_2) = T(u_2)$, all vertices in the set $I(w) = N_{k-m}(w) \cap T(v_2)$ are necessarily repeats of the vertex v. That is, $I(w) \subset R(v)$, and therefore $|I(w)| \leqslant 2$. Combining the two inequalities for $|I(w)|$ we have $2^{k-m} - 3 \leqslant |I(w)| \leqslant 2$, which immediately shows that $m = k-2$ or $m = k-1$. It remains to exclude the first possibility, which will be done using stability.

If $w \notin N_1(v_1) \cup \{v_1\}$ then, as w is a stable vertex, we have $|I(w)| = 0$, and so $2^{k-m} - 3 \leqslant 0$, which can only hold when $m = k-1$. Next, let $w \in N_1(v_1)$ and assume that $m = k-2$. Then $N_{k-m}(w) = N_2(w)$, and so (recalling that $k \geqslant 6$ throughout) the vertex u cannot be in $N_2(w)$. In this case we therefore have $I(w) \supset N_{k-m}(w)\setminus R(u_1)$, and so $|I(w)| \geqslant 2^{k-m} - 2 = 2^2 - 2 = 2$. But by stability of w and by the inclusion $I(w) \subset R(v)$ we have $|I(w)| \leqslant 1$, a contradiction. Finally, if $w = v_1$ and $m = k-2$ we again have $u \notin N_2(w)$. Moreover, due to the fact that $v \in N_{k-2}(u_1) \cup N_{k-1}(u_1)$, the vertex v_1 (now assumed to be in $N_{k-2}(u_1)$ as $m = k-2$ and $\ell = 0$) is a repeat of u_1. Looking at $N_2(w)$ ($w = v_1$) as a subset of $T(v_1)$ we see that $v_1 \notin N_2(w)$ in this case. Thus, if $w=v_1$ then the set $N_2(w)$ contains at most one vertex from $T(u_1)\cup\{u\}$, namely, the second repeat of u_1. It follows that for $w=v_1$ and $m=k-2$ we have $I(w)\supset N_{k-m}(w)\setminus(R(u_1)\setminus\{v_1\})$, which gives $|I(w)|\geqslant 2^{k-m}-1=2^2-1=3$, contrary to the fact that $I(w)\subset R(v)$. □

The above result together with Lemma 1 immediately implies the following fact.

Corollary 5. *If the vertex v_1 appears in $T(u_1)$, then $v_1\in N_{k-1}(u_1)$.*

Another application of Lemma 4 leads to a partial identification of repeats of the vertex v.

Lemma 6. *At least one of v_1, v_3, v_4 is a repeat of the vertex v.*

Proof. The claim is clearly equivalent to the assertion that at least one of the vertices v_1, v_3, v_4 is in $T(v_2)=T(u_2)$. Suppose on the contrary that this is not the case. As the vertex set of our $(2,k)$-digraph of defect 2 is $\{u\}\cup T(u_1)\cup T(u_2)$ and none of v_1, v_3, v_4 can coincide with u, it follows that $\{v_1,v_3,v_4\}\subset T(u_1)$. By Corollary 5 we then have $v_1\in N_{k-1}(u_1)$. As v_3 and v_4 are neighbours of $v_1\in N_{k-1}(u_1)$ and, at the same time, by our assumption the vertices v_3 and v_4 occur in $T(u_1)$, we conclude that $R(u_1)=\{v_3,v_4\}$.

We already know that the vertex v must appear in $T(u_1)$, more exactly, $v\in N_{k-2}(u_1)\cup N_{k-1}(u_1)$. But if $v\in N_{k-1}(u_1)$ then its neighbour v_1 would be a third repeat of u_1, which is impossible. Therefore $v\in N_{k-2}(u_1)$. It follows that, together with v_1, the second neighbour v_2 of the vertex v also appears in the set $N_{k-1}(u_1)$, and therefore $v_2\in R(u)$.

Now, let w_3 and w_4 be the in-neighbours of v_3 and v_4, respectively, such that $w_3,w_4\neq v_1$; note that by Lemma 1 we have $w_3\neq w_4$. From the fact that $v_1\in N_{k-1}(u_1)$ it follows that v_1 cannot appear anywhere else in $T(u_1)$, and so v_3 and v_4 can appear in $T(u_1)$ only as neighbours of w_3 and w_4, that is, $w_3,w_4\in T(u_1)\cup\{u\}$.

To finish our argument we look at the vertex u_2. Since the diameter of our digraph is k, we have $d(u_2,v_3)\leqslant k$ and $d(u_2,v_4)\leqslant k$. But according to our assumption, neither v_3 nor v_4 are in $T(u_2)$ and therefore $d(u_2,v_3)=d(u_2,v_4)=k$. As v_1 is not in $T(u_2)$ either, the only possibility to reach v_3 and v_4 from the vertex u_2 by a path of length k is that the other two in-neighbours of v_3 and v_4 are in $T(u_2)$; more exactly, we must have $w_3,w_4\in N_{k-1}(u_2)$. Then, however, the above paragraph implies that both w_3 and w_4 are repeats of u. This together with $v_2\in R(u)$ gives too many repeats from u (as clearly $w_3,w_4\neq v_2$), a contradiction. □

From the definition of stable vertices we now immediately obtain:

Corollary 7. *The vertices v_3 and v_4 are stable.*

We continue with a result that will help us locate the occurence of stable vertices from $T(v_1)$ in $T(u_1)$.

Lemma 8. *Among all the stable vertices $w \in T(v_1)$ such that $1 \leqslant d(v_1, w) \leqslant k-4$, at most one can appear in $N_{k-1}(u_1)$.*

Proof. Suppose that w and w' are two stable vertices of $T(v_1)$ such that $1 \leqslant d(v_1, w) \leqslant d(v_1, w') \leqslant k-4$ and such that $w, w' \in N_{k-1}(u_1)$. Looking at $T(v_1)$ it is clear that the neighbourhoods $N_1(w)$ and $N_1(w')$ must be disjoint and cannot contain the vertex u. Out of the four vertices in $N_1(w) \cup N_1(w')$, at most two can be repeats of u_1, so at least two of them must appear in $T(u_2) = T(v_2)$. However, vertices of $N_1(w) \cup N_1(w')$ that appear also in $T(v_2)$ are necessarily repeats of v, and hence $N_1(w) \cup N_1(w') = R(u_1) \cup R(v)$. It follows that exactly two vertices in $N_1(w) \cup N_1(w')$ are repeats of v_1. If $d(v_1, w) \geqslant 2$ or if $1 = d(v_1, w) < d(v_1, w')$ this already contradicts the stability of w and w'.

It remains to consider the case when $\{w, w'\} = N_1(v_1)$, and now the analysis depends on the position of v_1. If $v_1 \in T(u_1)$ then, by Corollary 5, the vertex v_1 belongs to $N_{k-1}(u_1)$. Due to the assumption that $\{v_3, v_4\} = \{w, w'\} \subset N_{k-1}(u_1)$ we then have $R(u_1) = \{v_3, v_4\}$. But then no vertex in $N_1(w) \cup N_1(w')$ can be a repeat of u_1, contrary to the fact that $N_1(w) \cup N_1(w') = R(u_1) \cup R(v)$. Finally, if $v_1 \in T(u_2)$ then $v_1 \in R(v)$ and, as clearly $v_1 \notin N_1(w) \cup N_1(w')$ we again have a contradiction with $N_1(w) \cup N_1(w') = R(u_1) \cup R(v)$. □

In all statements and arguments of this section the roles of the vertices u and v can obviously be interchanged. For example, a vertex $w \in T(u_1)$ is stable if either $w = u_1$, or $w \in N_1(u_1)$ and there is at most one repeat of u in the set $T_{k-2}(w) \setminus \{w\}$, or else if $w \in N_\ell(u_1)$ for $2 \leqslant \ell \leqslant k-3$ and the set $T_{k-1-\ell}(w) \setminus \{w\}$ contains no repeat of u. As regards Lemmas 4–8 and Corollaries 5 and 7, replacing u with v and u_i with the corresponding v_i (and vice versa) we obtain their 'dual' versions; for example, it follows that u_3 and u_4 are stable vertices. Rather than stating each of these 'dual' results separately we present the following obvious consequence of the 'primal' and 'dual' versions of Lemmas 4 and 8.

Proposition 9. *Let W_u and W_v be the sets of all stable vertices in $T(u_1)$ and $T(v_1)$ that are not a repeat of u and v and such that their distance from u_1 and v_1, respectively, is at least 1 and at most $k-4$. Then, $W_u \subset T(v_1)$ and $W_v \subset T(u_1)$. Moreover, there is at most one vertex $w_u \in W_u$ and $w_v \in W_v$ such that $w_u \in N_{k-1}(v_1)$ and $w_v \in N_{k-1}(u_1)$; for all the remaining vertices $w \in W_u \cup W_v \setminus \{w_u, w_v\}$ we have $d(u_1, w) = d(v_1, w)$.*

4. The main result

In this section we prove our main result about the non-existence of $(2, k)$-digraphs of defect two if $k \geqslant 6$. We will do this by systematically examining the possibilities for $R(v)$, the set of repeats of the vertex v. Again, we will consider two fixed vertices u and v of our $(2,k)$-digraph H of defect two, as depicted in Fig. 2; we also keep referring to the standard notation introduced therein.

Lemma 10. *At least one of the vertices v_1, v_3, v_4 is not a repeat of v.*

Proof. On the contrary, assume that $R(v) \subset \{v_1, v_3, v_4\}$. As v_1, v_3 and v_4 are all in $T(v_1)$, our assumption means that at least two of these vertices are in $T(v_2) = T(u_2)$ as well. Let $N_1(v_3) = \{v_7, v_8\}$ and $N_1(v_4) = \{v_9, v_{10}\}$. It is important to observe that, in this case, all the vertices v_7, v_8, v_9, v_{10} are stable. In the '$T(u_1)$-counterpart', we similarly let $N_1(u_3) = \{u_7, u_8\}$ and $N_1(u_4) = \{u_9, u_{10}\}$.

Now, from Lemma 2 we see that $\{u_7, u_8, u_9, u_{10}\} \neq \{v_7, v_8, v_9, v_{10}\}$; we may assume that $v_7 \notin \{u_7, u_8, u_9, u_{10}\}$. But then, since *all* the v_i mentioned above are stable vertices, by Proposition 9, we conclude that $v_7 \in N_{k-1}(u_1)$ and, without loss of generality, $u_i = v_i$ for $i = 8, 9, 10$. Moreover, as the vertex v_7 cannot appear in the sets $T_{k-3}(v_i)$ for $i = 8, 9, 10$ and $v_i = u_i$ for these i, the vertex v_7 must appear in $N_{k-1}(u_1)$ so that it is contained in the set $T_{k-3}(u_7)$.

To conclude the argument we consider the neighbours of the vertex v_7 in $T(v_1)$. If $N_1(v_7) \subset \{u_1, u_3, u_4\}$ then the vertices of $N_1(v_7)$ are stable and, by Proposition 9, at least one of them would have to be at a distance $\leqslant 1$ from v_1, which is clearly impossible. Thus, there is at least one vertex, say, $w \in N_1(v_7)$ such that $w \notin \{u_1, u_3, u_4\}$.

Since w can appear only once in $T_{k-1}(v_1)$ and it appears there in $T_{k-3}(v_7)$, it follows that $w \notin T_{k-3}(v_i)$ for $i = 8, 9, 10$. By our assumption that $R(v) \subset \{v_1, v_3, v_4\}$, the vertex w cannot be a repeat of v. Hence, $w \notin T(v_2) = T(u_2)$, and therefore w must appear in $\{u\} \cup T(u_1)$. Clearly $w \neq u$ and, by the above, $d(u_1, w) \geqslant 3$. Also, we have established before that $u_i = v_i$ for $i = 8, 9, 10$ and therefore $w \notin T_{k-3}(u_i)$ for $i = 8, 9, 10$. Therefore, the only possibility for w is that $w \in T_{k-3}(u_7)$. But as it was shown two paragraphs ago, v_7 is in $T_{k-3}(u_7)$ as well, and since $w \in N_1(v_7)$ we thus have two short $u_7 \to w$ paths. This contradiction completes the proof. □

In view of Lemma 6 and the preceding result we have the following obvious corollary.

Corollary 11. *Exactly one of the vertices v_1, v_3, v_4 is a repeat of v.*

We first exclude the possibility that $v_1 \in R(v)$ and then try to locate the vertex v_4 in $T(u_1)$ under the assumption that $v_4 \in R(v)$.

Lemma 12. *The vertex v_1 is not a repeat of v.*

Proof. Let $v_1 \in R(v)$. By Lemma 10 we see that neither v_3 nor v_4 can be in $T(v_2) = T(u_2)$, and therefore $v_3, v_4 \in T(u_1)$. If, say, $v_3 = u_1$ then, by the '$T(v_1)$-version' of Corollary 5 we would have $u_1 = v_3 \in N_{k-1}(v_1)$ and hence two short $v_1 \to v_3$ paths.

This shows that $v_3, v_4 \neq u_1$. Invoking Lemma 1 we then may assume without loss of generality that $v_3 \notin \{u_1\} \cup N_1(u_1)$. But then, as v_3 is a stable vertex, Lemma 4 shows that $v_3 \in N_{k-1}(u_1)$. Applying now Proposition 9 it follows that $v_4 \in N_1(u_1)$; we may assume that $u_4 = v_4$. Of course, the vertex v_3 appears in $N_{k-1}(u_1)$ in such a way that $v_3 \in N_{k-2}(u_3)$.

Again, let $N_1(v_3 = \{v_7, v_8\}$; note that $v_3, v_7, v_8 \notin T_{k-2}(v_4)$. Clearly, at most one vertex of $N_1(v_3)$ can be in $T(v_2)$, otherwise (together with the vertex v_1) we would have three repeats of v. Consequently, at least one vertex of $N_1(v_3)$, say, v_7, has to appear in $\{u\} \cup T(u_1)$. It is easy to see that $v_7 \neq u, u_1$, and as $u_4 = v_4$, we also have $v_7 \notin T_{k-2}(u_4)$. Thus, v_7 must belong to the set $T(u_3)$. However, we also have $v_3 \in N_{k-2}(u_3)$, and this gives rise to two short $u_3 \to v_7$ paths, a contradiction. □

It follows that $R(v)$ contains exactly one of the vertices v_3, v_4 and, of course, $v_1 \notin R(v)$. We assume without loss of generality that $v_4 \in R(v)$ and prove Lemma 13.

Lemma 13. *Let v_4 be a repeat of the vertex v. Then, $v_3 \in N_1(u_1)$.*

Proof. Since by Lemma 12 we have $v_1 \notin R(v)$, it follows that $v_1 \in T(u_1)$ and from Corollary 5 we see that $v_1 \in N_{k-1}(u_1)$. The vertex v_3 is stable and, by our assumption combined with Lemma 10, v_3 appears in $T(u_1)$. According to Proposition 9 we then have either $v_3 \in N_1(u_1)$ or $v_3 \in N_{k-1}(u_1)$. To prove the lemma it remains to exclude the second possibility. But in any case, since v_3 is a neighbour of $v_1 \in N_{k-1}(u_1)$, it follows that $v_3 \in R(u_1)$.

Suppose that $v_3 \in N_{k-1}(u_1)$. As before, let $N_1(v_3) = \{v_7, v_8\}$ and $N_1(v_4) = \{v_9, v_{10}\}$. Clearly, the set $N_1(v_3)$ cannot contain any of the vertices u and v, and therefore $N_1(v_3) \subset T(u_1) \cup T(u_2)$. We cannot have $N_1(v_3) \subset T(u_1)$, as this would mean that the vertices in $N_1(v_3)$ are also repeats of u_1, contrary to the fact that $v_3 \in R(u_1)$. On the other hand, we cannot have $N_1(v_3) \subset T(u_2) = T(v_2)$, because then the vertices in $N_1(v_3)$ would be another two repeats of v, contrary to the assumption of our lemma. Thus, without loss of generality, $v_7 \in T(u_1)$ and $v_8 \in T(u_2)$, that is, $R(u_1) = \{v_3, v_7\}$ and $R(v) = \{v_4, v_8\}$.

It is now clear that v_7, v_9 and v_{10} are stable vertices that are not repeats of v (and, of course, so is v_3). Since $v_3 \in N_{k-1}(u_1)$, Proposition 9 shows that the vertices v_7, v_9, v_{10} must all belong to $N_2(u_1)$. But then, by Lemma 1, the vertex v_4 must appear in $N_1(u_1)$. Because of the fact that v_4 is also a neighbour of the vertex $v_1 \in N_{k-1}(u_1)$, it follows that v_4 is a repeat of u_1, which contradicts the fact that $R(u_1) = \{v_3, v_7\}$. □

We are finally ready to prove the main result of this paper.

Theorem 14. *There are no $(2,k)$-digraphs of defect two if $k \geqslant 3$.*

Proof. In view of the facts mentioned in Section 2, we may assume that $k \geqslant 6$. By all the previous auxiliary results we may assume that if a $(2,k)$-digraph of defect two

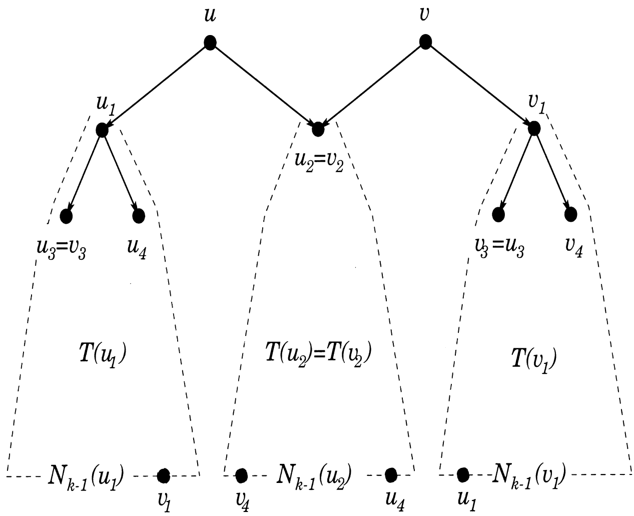

Fig. 3. Illustration for the proof of Theorem 1.

exists, it has the structure as depicted in Fig. 2. Moreover, we may assume that v_4 is a repeat of v whereas $v_1, v_3 \notin R(v)$, and that $v_3 \in N_1(u_1)$; without loss of generality, $v_3 = u_3$. Invoking $u - v$ and $u_i - v_i$, the symmetry of our situation and of all the previous statements and their proofs, we may also assume that $u_4 \in R(u)$ but $u_1, u_3 \notin R(u)$. As $u_1 \notin R(u)$ and $v_1 \notin R(v)$, we must have $u_1 \in T(v_1)$ and $v_1 \in T(u_1)$. Hence, according to Lemma 5 we have $u_1 \in N_{k-1}(v_1)$ and $v_1 \in N_{k-1}(u_1)$. Also, note that both u_4 and v_4 must appear in $T(u_2) = T(v_2)$ in the 'bottom level', that is, in $N_{k-1}(u_2)$. The situation is depicted in Fig. 3, and the reader is invited to draw in the steps that follow.

Since now the vertex $u_3 = v_3$ is not in $T(u_2)$, it follows that $d(u_2, u_3) = k$. Pick a $u_2 \to u_3$ path of length k; the immediate predecessor of u_3 in this path must appear in $N_{k-1}(u_2)$. However, u_3 has only two possible in-neighbours, namely, u_1 and v_1; we therefore may assume that, say, $u_1 \in N_{k-1}(u_2)$. But this implies that u_1 is another repeat of u, so that $R(u) = \{u_1, u_4\}$. By Lemma 3 we know that $v \in N_{k-2}(u_1) \cup N_{k-1}(u_1)$. If $v \in N_{k-2}(u_1)$ then we would also have $v_2 = u_2 \in R(u)$, a contradiction. Therefore, v must be in the set $N_{k-1}(u_1)$, which implies that for the in-neighbour $w \in T(u_1)$ of the vertex $v_1 \in N_{k-1}(u_1)$ we have $w \neq v$. Using the fact that $v_1 \notin T(u_2)$ we infer that $d(u_2, v_1) = k$. As above, take a $u_2 \to v_1$ path of length k in our digraph; the immediate predecessor of v_1 in the path has to appear in $N_{k-1}(u_1)$. The only two candidates for this predecessor are v and w, and v is clearly excluded. It follows that $w \in N_{k-1}(u_2)$, which shows that w is a repeat of u. But by the above analysis, $R(u) = \{u_1, u_4\}$ and we clearly have $w \neq u_1, u_4$. This final contradiction proves the theorem. □

Thus, for the defect we have $\varepsilon_{2,k} \geqslant 3$ if $k \geqslant 3$. As the exact value of the defect seems to be extremely difficult to determine, it would be of high interest to find at least good asymptotic bounds on $\varepsilon_{2,k}$ for all $k \geqslant 3$.

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